FINAL EXAM

ALGEBRAIC STRUCTURES (2022/23)

19/06/2023

The maximum number of points you can score in this Evaluation is 10. You get 1 point for free.

Instructions: Read the questions carefully and take your time to access which part of the theory you need for each exercise. **All your answers should be accompanied by a justification.**

Good luck!

Exercise 1. [2 points]

a) [1 point] *Give a prime factorization of* $\beta = 310 + 310i$ *in* $\mathbb{Z}[i]$.

b) [1 point] *Is the polynomial* $x^{310} - (310 + 310i)$ *irreducible over* $\mathbb{Q}[i]$*?*

Don't forget to justify your answers.

Exercise 2. [2.5 points] *Let* $\omega = \sqrt{3} + \sqrt{5}$.

a) [1 point] *Determine the minimal polynomial of* ω *over* $\mathbb{Q}$.

b) [0.5 point] *Determine* $[\mathbb{Q}(\omega) : \mathbb{Q}]$.

c) [1 point] *Is* $\mathbb{Q}(\omega)$ *a splitting field for the polynomial determined in a)?*

Don't forget to justify your answers.

Exercise 3. [2.3 points]

a) [1 point] *Determine all monic irreducible polynomials* $f \in \mathbb{F}_3[X]$ *for which* $\mathbb{F}_3[X]/(f) \simeq \mathbb{F}_9$.

b) [0.7 point] *Let* f *be one of the polynomials determined in a). Show that* f *does not divide* $X^{27} - X$.

c) [0.6 point] *Show that there is no field with 27 elements that contains* $\mathbb{F}_9$.

Don't forget to justify your answers.

Exercise 4. [2.2 points] *Let* $i \in \mathbb{C}$ *be such that* $i^2 = -1$ *and* $\mathbb{Z}[i\sqrt{5}] = \{a + bi\sqrt{5};\, a, b \in \mathbb{Z}\}$. *Consider the evaluation homomorphism* $f : \mathbb{Z}[X] \to \mathbb{Z}[i\sqrt{5}]$ *given by* $f(X) = i\sqrt{5}$ *and* $f(a) = a, \forall a \in \mathbb{Z}$.

a) [0.7 point] *Show that* f *is surjective and determine its kernel.*

b) [1.5 point] *Show that the ideal* $(2, 1 + i\sqrt{5})$ *is a maximal ideal in* $\mathbb{Z}[i\sqrt{5}]$ *(Hint: You can use the homomorphism* f *to show that* $(\mathbb{Z}[i\sqrt{5}]/(2, 1 + i\sqrt{5}) \cong \mathbb{F}_2$.*)*

Don't forget to justify your answers.